Transit Network Design Problem:

A mathematical formulation and heuristic solution

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Content

- Introduction and previous work
- Definitions and hypothesis
- Mathematical programming formulation
- Heuristic solution and some preliminary numerical results
- Conclusions and future work
Introduction and previous work
Optimization of routes and frequencies

- **Actors:**
  - Users
  - Operator
  - Regulator (transit agency, municipality)

- **Objectives:**
  - Minimize travel time (access, on-board and waiting) of the users
  - Minimize operation costs

- **Decision variables:**
  - Routes, in terms of the street network
  - Frequency: buses per hour on the routes

- **Data:**
  - Street network
  - Demand (origin-destination matrix)
Previous work

- Mathematical programming formulations:
  - Schöbel and Scholl (2006): Minimize on-board travel time and transfers under bus capacity and budgetary constraints.
  - Borndörfer et al (2007): Minimize on-board travel time and operator’s cost under bus and street capacity constraints. Feasible solutions and lower bounds are provided for a real case.
  - Waiting time at stops and controled transfers are not considered at the same time.

- Heuristics without an underlying explicit formulation.
Definitions and hypothesis
Definitions and hypothesis

- An undirected graph $G^U = (V,E)$ models the underlying network; all of its vertices are centroids (originators of demand), bus stops and street nodes at the same time.

- A set $K$ of origin-destination pairs models the demand, each $k \in K$ has:
  - Origin and destination vertices $O_k$ and $D_k$ respectively.
  - A demand quantity $\delta_k$ (trips per time unit) in a given time horizon.

- Each route is a (loopless) sequence of adjacent vertices in $G^U$; forward and backward directions of the route are symmetrical and they have the same duration.

- $R$ is the set of all possible routes defined as above.

- The frequencies are real positive numbers.
Behavior of passengers

- Given an OD pair \( k \in K \) and a set of routes with frequencies:
  - Which route or combination of routes will be used by the users for traveling from the origin to the destination?
  - What information do they take into account?
  - Do all users will use the same routes?
  - How do they behave if there is no sufficient capacity in the routes that they want to use?

- Assignment model: a descriptive model that in this case is embedded into a prescriptive (optimization) model.
Behavior of passengers

- Assignment model of Spiess and Florian (1989), optimal strategies.
  - Which route or combination of routes will be used by the users for traveling from the origin to the destination?

  The combination of routes that minimizes the access and on-board travel time plus the expected waiting time at the stop(s).
Behavior of passengers

- Assignment model of Spiess and Florian (1989), optimal strategies.

  - Which route or combination of routes will be used by the users for traveling from the origin to the destination?

    The combination of routes that minimizes the access and on-board travel time plus the expected waiting time at the stop(s).

  - What information do they take into account?

    Before arriving to the bus stop: routes and their frequencies, access and on-board travel time. They select a set of attractive routes.

    While waiting at the stop: buses passing by that stop, belonging to the set of attractive routes identified a priori. They take the first bus.
Behavior of passengers

- Assignment model of Spiess and Florian (1989), optimal strategies.
  - Do all users will use the same routes?
    - The passengers corresponding to the same OD pair are distributed among the attractive routes proportional to their frequencies.
Behavior of passengers

- Assignment model of Spiess and Florian (1989), optimal strategies.
  - Do all users will use the same routes?

    The passengers corresponding to the same OD pair are distributed among the attractive routes proportional to their frequencies.
  - How do they behave if there is no sufficient capacity in the routes that they want to use?

    Sufficient capacity is assumed; it should be ensured by the optimization model (user equilibrium is not modeled).
Mathematical programming formulation
Network model

- Multi-commodity network flow model.

- Routes are defined over $G^U$.

- A given set of routes induces a directed graph $G^T = (N, A)$ which is used to define the trajectories of the passengers from the origin to the destination.
Network model

Graph $G^U$, one OD pair

![Diagram of a network with two OD pairs connected by edges]
Network model

Graph $G^U$, one OD pair and three routes
Network model

Graph $G^U$, one OD pair and three routes

Graph $G^T$ induced

- **On-board travel arc**
- **Wait arc**
- **Destination (egress) arc**
Optimal strategies

- If $A^+_n$ is the set of outgoing arcs of node $n$, corresponding to the attractive routes of the strategy, then (assuming uniform and Poisson arrival of passengers and buses respectively to the bus stop, $\alpha = 1$):

  - Expected waiting time at node $n$: $\frac{\alpha}{\sum_{a \in A^+_n} f_{r_a}}$

  - Proportion of demand that uses the outgoing arc $a$: $\frac{f_{r_a}}{\sum_{a' \in A^+_n} f_{r_{a'}}}$
Optimal strategies

Formulation of the assignment problem for a single OD pair

\[
\begin{align*}
\min_{v,v',x} & \quad \sum_{a \in A} C_a v_a + \sum_{n \in N} \frac{V_n}{\sum_{a \in A_n^+} f_a x_a} \\
\text{s.t.} & \quad V_n = \sum_{a \in A_n^-} v_a + g_n \quad \forall n \in N \\
& \quad v_a = \frac{x_a f_a}{\sum_{a' \in A_n^+} f_{a'} x_{a'}} V_n \quad \forall a \in A_n^{E+}, n \in N \\
& \quad V_n \geq 0 \quad \forall n \in N \\
& \quad x_a \in \{0,1\} \quad \forall a \in A
\end{align*}
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Optimal strategies

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& \quad v_a = \frac{x_a f_a}{\sum_{a' \in A_n^+} f_a x_a}, \quad \forall a \in A_n^{E+}, n \in N \\
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Non-linear objective function
Optimal strategies

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Formulation of the assignment problem for a single OD pair

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s.t. \( V_n = \sum_{a \in A_n^-} v_a + g_n \quad \forall n \in N \)

\[
v_a = \frac{x_a f_a}{\sum_{a' \in A_n^+} f_{a'} x_{a'}} V_n \quad \forall a \in A_n^{E+}, n \in N
\]

\( V_n \geq 0 \quad \forall n \in N \)

\[
x_a \in \{0,1\} \quad \forall a \in A
\]
Optimal strategies

Change of variables and elimination of binary variables: linear formulation

\[
\min_{v,w} \sum_{a \in A} c_a v_a + \sum_{n \in N} w_n
\]

s.t.
\[
\sum_{a \in A_n^+} v_a - \sum_{a \in A_n^-} v_a = b_n \quad \forall n \in N
\]

\[
v_a \leq f_{r_a} w_n \quad \forall a \in A_{n}^{E+}, n \in N
\]

\[
v_a \geq 0 \quad \forall a \in A
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Optimal strategies

Change of variables and elimination of binary variables: linear formulation

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\end{align*}
\]

Splitting of flow

\[
v_a \leq f_{r_a} w_n \quad \forall a \in A_n^{E+}, n \in N
\]

\[
v_a \geq 0 \quad \forall a \in A
\]

Shortest hyper-path (Nguyen and Pallotinno, 1988).
Controling the transfers

- Maximum number of allowed transfers, parameter given a priori: $\tau^m$

- Minimum proportion of the total demand $\sum_{k \in K} \delta_k$ that travels with no more than a given number of transfers: $0 \leq \Delta^0 \leq \ldots \leq \Delta^\tau^m \leq 1$
Controlling the transfers

One graph $G^T_i$ for each stage of travel (previous example for $\tau^m=1$)
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Controlling the transfers

One graph $G^T_i$ for each stage of travel (previous example for $\tau^m=1$)
Optimization of routes and frequencies

\[
\begin{align*}
\text{min} & \quad \text{(users' interests)} \\
\text{s.t.} & \quad \text{operator's interests} \\
& \quad \text{capacities of the streets} \\
& \quad \text{capacities of the buses} \\
& \quad \text{proportions of demand performing transfers} \\
& \quad \text{behavior of the passengers}
\end{align*}
\]
\[
\min_{x,f,v,w} \sum_{k \in K} \left( \sum_{a \in A} c_a v_{ak} + \sum_{n \in N} w_{nk} \right)
\]

s.t.
\[
\sum_{r \in R} f_r \sum_{a \in A_r} c_a \leq B
\]
\[
\sum_{a \in A_e} f_{ra} \leq \kappa_e \quad e \in E
\]
\[
\sum_{k \in K} v_{ak} \leq f_{ra} \omega^b \quad a \in A
\]
\[
\sum_{k \in K} \sum_{a \in A_1^D \cup \ldots \cup A_{s+1}^D} v_{ak} / \sum_{k \in K} \delta_k \geq \Delta^s \quad s \in [0..\tau^m - 1]
\]
\[
x_r \in \{0,1\} \quad r \in R
\]
\[
0 \leq f_r \leq f^m x_r \quad r \in R
\]
\[ \sum_{a \in A_n^+} v_{ak} - \sum_{a \in A_n^-} v_{ak} = b_{nk} \quad n \in N, k \in K \]

\[ v_{ak} \leq f_{r_a} w_{nk} \quad n \in N, a \in A_n^{E+} \]

\[ 0 \leq v_{ak} \leq \delta_k x_{r_a} \quad a \in A, k \in K \]

\[ b_{nk} = \begin{cases} 
\delta_k & \text{if } n = O_k \\
-\delta_k & \text{if } n = D_k \\
0 & \text{otherwise}
\end{cases} \]
Optimization of routes and frequencies

- The objective function models simultaneously the interests of the users and its behavior.

- Actually, decisions are taken by different agents:
  - Interests: planner of the system, decision maker (variables $x$ and $f$).
  - Behavior: users by themselves (variables $v$ and $w$).

- In the proposed formulation, the demand is assigned under constraints of bus capacity: the produced flows will not be necessarily consistent with the hypothesis of the assignment model (which ignores bus capacities).

- Adequate framework to model this situation: mathematical programming in two levels (or bilevel, Bard, 1998).
Bilevel mathematical programming

\[
\begin{align*}
\min_{x, y} & \quad F(x, y) \\
\text{s.t.} & \quad G(x, y) \leq 0 \\
& \quad y \in \arg\min_y f(x, y) \\
\text{s.t.} & \quad g(x, y) \leq 0
\end{align*}
\]

Upper level: takes decisions \(x\), anticipating the reaction \(y\) of the lower level.

Lower level: takes decisions \(y\), subject to decisions \(x\) of the upper level.
\[
\min_{x,f,v,w} \sum_{k \in K} \left( \sum_{a \in A} c_a v_{ak} + \sum_{n \in N} w_{nk} \right)
\]
s.t. \[
\sum_{r \in R} f_r \sum_{a \in A_r} c_a \leq B
\]

\[
\sum_{a \in A_e} f_{r_a} \leq \kappa_e \quad e \in E
\]

\[
x_r \in \{0,1\} \quad r \in R
\]

\[
0 \leq f_r \leq f^m x_r \quad r \in R
\]

\[
\min_{v,w} \sum_{k \in K} \left( \sum_{a \in A} c_a v_{ak} + \sum_{n \in N} w_{nk} \right)
\]
s.t. \[
\sum_{a \in A^+_n} v_{ak} - \sum_{a \in A^-_n} v_{ak} = b_{nk} \quad n \in N, k \in K
\]

\[
v_{ak} \leq f_{r_a} w_{nk} \quad n \in N, a \in A^+_n
\]

\[
b_{nk} = \begin{cases} 
\delta_k & \text{if } n = O_k \\
-\delta_k & \text{if } n = D_k \\
0 & \text{otherwise}
\end{cases} \]
\[ \sum_{k \in K} v_{ak} \leq f_{ra} \omega^b \quad a \in A \]

\[ \sum_{k \in K} \sum_{a \in A^D_1 \cup \ldots \cup A^D_{s+1}} v_{ak} = \sum_{k \in K} \delta_k \geq \Delta^s \quad s \in [0..\tau^m - 1] \]

A similar formulation is proposed by Constantin and Florian (1996) for optimizing frequencies.
\[
\min_{x,f,v,w} \sum_{k \in K} \left( \sum_{a \in A} c_a v_{ak} + \sum_{n \in N} w_{nk} \right)
\]

s.t. \[
\sum_{r \in R} f_r \sum_{a \in A_r} c_a \leq B
\]
\[
\sum_{a \in A_e} f_{ra} \leq \kappa_e \quad e \in E
\]
\[
x_r \in \{0,1\} \quad r \in R
\]
\[
0 \leq f_r \leq f^m x_r \quad r \in R
\]

\[
\min_{v,w} \sum_{k \in K} \left( \sum_{a \in A} c_a v_{ak} + \sum_{n \in N} w_{nk} \right)
\]

s.t. \[
\sum_{a \in A_n^+} v_{ak} - \sum_{a \in A_n^-} v_{ak} = b_{nk} \quad n \in N, k \in K
\]
\[
v_{ak} \leq f_{ra} w_{nk} \quad n \in N, a \in A_n^{E^+}
\]
\[
b_{nk} = \begin{cases} 
\delta_k & \text{if } n = O_k \\
-\delta_k & \text{if } n = D_k \\
0 & \text{otherwise}
\end{cases}
\]
\[
0 \leq v_{ak} \leq \delta_k x_{ra} \quad a \in A, k \in K
\]
\[
\min_{x,f,v,w} \sum_{k \in K} \left( \sum_{a \in A} c_{ak} v_{ak} + \sum_{n \in N} w_{nk} \right)
\]
s.t. \[
\sum_{r \in R} f_{rv} \sum_{a \in A_r} c_{a} \leq B
\]
\[
\sum_{a \in A_e} f_{rv} \leq \kappa_e, \quad e \in E
\]
\[
x_r \in \{0,1\}, \quad r \in R
\]
\[
0 \leq f_r \leq f^m x_r, \quad r \in R
\]

\[
\min_{v,w} \sum_{k \in K} \left( \sum_{a \in A} c_{ak} v_{ak} + \sum_{n \in N} w_{nk} \right)
\]
s.t. \[
\sum_{a \in A_n^+} v_{ak} - \sum_{a \in A_n^-} v_{ak} = b_{nk}, \quad n \in N, k \in K
\]
\[
v_{ak} \leq f_{rv} w_{nk}, \quad n \in N, a \in A_n^{E+}
\]
\[
b_{nk} = \begin{cases} 
\delta_k & \text{if } n = O_k \\
-\delta_k & \text{if } n = D_k \\
0 & \text{otherwise}
\end{cases}
\]
\[
0 \leq v_{ak} \leq \delta_k x_r, \quad a \in A, k \in K
\]
Constraints of the upper level that depends on variables of both levels or the lower level only: decisions of the planner, who needs to know the reaction of the users with respect to his decisions.
Difficulties of resolution

- Bilevel nature.

- Exponential number of discrete variables $x$.

- Non-linear constraint.

$$v_{ak} \leq f_r w_{nk} \quad n \in N, a \in A^E_n$$
Difficulties of resolution, non-linear constraint

- Frequencies are taken from a predetermined discrete set:
  \[ F = \{ \phi_1, \ldots, \phi_{|F|} \} \], indexed by \( f \).

- The continuous variable \( f_r \) is substituted by the binary variable \( y_{rf} \) which takes value 1 if the route \( r \) has the frequency \( f \).

- The equations that involve variable \( f \) should be rewritten in terms of \( y \).

- The (formerly) non-linear constraint now can be written as:
  \[ v_{ak} \leq \phi_{fa} w_{nk} \quad n \in N, a \in A_n^{E^+} \]
  where \( f_a \) is the index in \( F \) corresponding to the arc \( a \).
Difficulties of resolution, non-linear constraint

- The graph of trajectories should include a wait arc for each frequency of $F$.

- The resulting problem is a Discrete Continous Linear Bilevel problem, i.e. linear bilevel with discrete variables in the upper level and continuous variables in the lower level.
Difficulties of resolution

- Although general purpose exact solution methods for bilevel discrete linear problems are available, the exponential number of variables precludes the exact solution of realistic sized instances by using directly the mathematical formulation.

- The formulation is useful for obtaining lower bounds for small instances of the problem and for reasoning about the structure of the problem.

- For instances of realistic size (in terms of number of vertices, edges and OD pairs) we propose using approximate methods (heuristics).
Difficulties of resolution

- If the objective function of the lower level is suppressed and all the constraints are moved to the upper level: we obtain a relaxation of the original bilevel problem, which may be tackled using mixed integer linear programming techniques.

\[
\begin{align*}
\min_{x,y} & \quad F(x,y) \\
\text{s.t.} & \quad G(x,y) \leq 0 \\
& \quad y \in \arg \min_y f(x,y) \\
& \quad g(x,y) \leq 0
\end{align*}
\]

- The objective functions of both levels are the same: for very high bus capacities and no allowed transfers, the optimal solution of the relaxation is the same as the one of the original bilevel problem.
Heuristic solution and some preliminary numerical results
Heuristic solution

- A simple heuristic.

- It solves approximately the problem without street and bus capacities.

- A variant of the GRASP TNDP algorithm (Mauttone and Urquhart, 2006).
Test instance: small

- 8 vertices
- 10 edges
- 4 OD pairs
- 94 routes
- 3 frequencies

<table>
<thead>
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<th>Maximum fleet size</th>
<th>Demand covered directly</th>
<th>1.0 (no transfers)</th>
<th>0.5</th>
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</thead>
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<tr>
<td></td>
<td>Heur</td>
<td>LR</td>
<td>Gap</td>
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<td>271.39</td>
<td>16.07</td>
</tr>
<tr>
<td>20</td>
<td>277.50</td>
<td>244.84</td>
<td>13.34</td>
</tr>
<tr>
<td>60</td>
<td>256.67</td>
<td>236.34</td>
<td>8.60</td>
</tr>
</tbody>
</table>

* No feasible solution was found after 4 hours
Test instance: Wan and Lo

- 10 vertices
- 19 edges
- 8 OD pairs
- 411 routes (limit on route duration)
- 4 frequencies

<table>
<thead>
<tr>
<th>Demand covered directly</th>
<th>1.0 (no transfers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum fleet size</td>
<td>Heur</td>
</tr>
<tr>
<td></td>
<td>995</td>
</tr>
</tbody>
</table>
Conclusions and future work
Conclusions

- A mathematical programming formulation for the Transit Network Design Problem was presented.

- It considers the waiting time in the behavior of the passengers and allows to control the transfers.

- The size of the model grows very quickly according to the size of the instance.

- Main difficulties to solve exactly the formulation are:
  - The exponential number of variables that represent the routes.
  - The bilevel nature.
Future work

- Incorporate street and bus capacities to the heuristic.

- Develop a metaheuristic to improve the quality (objective value) of the solutions produced by the simple heuristic.

- Test with bigger and real instances.
Thank you
References